Laser Physics 7
Two-frequency lasers

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Outline

• Two-mode laser
  – Cross-saturation
  – Steady-state solutions
  – Transient behavior

• Injection locking
  – Rate equations
  – Adler’s equation
  – Locked and unlocked solutions
• Example 1: spatial hole burning spatial hole

• Example 2: inhomogeneous broadening

• Example 3: polarization
Dual-mode laser

- Consider a laser sustaining oscillation of two modes, labeled 1 and 2.
- If the two modes are independent, we have the following equations:

\[
\frac{dF_1}{dt} = -\frac{F_1}{\tau_{cav1}} + \kappa_1 F_1 \Delta N_1
\]

\[
\frac{d}{dt} \Delta N_1 = \frac{1}{\tau} \left( \Delta N_{01} - \Delta N_1 \right) - 2^* \kappa_1 F_1 \Delta N_1
\]

\[
\frac{dF_2}{dt} = -\frac{F_2}{\tau_{cav2}} + \kappa_2 F_2 \Delta N_2
\]

\[
\frac{d}{dt} \Delta N_2 = \frac{1}{\tau} \left( \Delta N_{02} - \Delta N_2 \right) - 2^* \kappa_2 F_2 \Delta N_2
\]
• However, in many situations (longitudinal modes, polarization modes, etc.), each mode saturates both population inversion reservoirs. One must introduce cross-saturation terms:

\[
\frac{dF_1}{dt} = -\frac{F_1}{\tau_{cav1}} + \kappa_1F_1 \Delta N_1
\]

\[
\frac{d}{dt} \Delta N_1 = \frac{1}{\tau} (\Delta N_{01} - \Delta N_1) - 2^*\kappa_1 \Delta N_1 (F_1 + \xi_{12}F_2)
\]

\[
\frac{dF_2}{dt} = -\frac{F_2}{\tau_{cav2}} + \kappa_2F_2 \Delta N_2
\]

\[
\frac{d}{dt} \Delta N_2 = \frac{1}{\tau} (\Delta N_{02} - \Delta N_2) - 2^*\kappa_2 \Delta N_2 (F_2 + \xi_{21}F_1)
\]
• For a class-A laser, we adiabatically eliminate $\Delta N_1$ and $\Delta N_2$:

\[
\frac{dF_1}{dt} = \frac{F_1}{\tau_{\text{cav}1}} \left[ -1 + \frac{r_1}{1 + (F_1 + \xi_{12}F_2)/F_{\text{sat}1}} \right]
\]

\[
\frac{dF_2}{dt} = \frac{F_2}{\tau_{\text{cav}2}} \left[ -1 + \frac{r_2}{1 + (F_2 + \xi_{21}F_1)/F_{\text{sat}2}} \right]
\]
Dual-mode class-A laser

\[
\begin{align*}
\frac{dF_1}{dt} &= \frac{F_1}{\tau_{\text{cav}1}} \left[ -1 + \frac{r_1}{1 + (F_1 + \xi_{12}F_2)/F_{\text{sat}1}} \right] \\
\frac{dF_2}{dt} &= \frac{F_2}{\tau_{\text{cav}2}} \left[ -1 + \frac{r_2}{1 + (F_2 + \xi_{21}F_1)/F_{\text{sat}2}} \right]
\end{align*}
\]

- 4 possible steady-state solutions:

Intersection between

\[
F_1 = 0 \text{ or } F_1 + \xi_{12}F_2 = F_{\text{sat}1}(r_1 - 1)
\]

AND

\[
F_2 = 0 \text{ or } F_2 + \xi_{21}F_1 = F_{\text{sat}2}(r_2 - 1)
\]

We suppose that \(r_1, r_2 > 1\)
Dual-mode class-A laser

\[
F_1 = 0 \text{ or } F_1 + \xi_{12} F_2 = F_{\text{sat1}}(r_1 - 1) \\
F_2 = 0 \text{ or } F_2 + \xi_{21} F_1 = F_{\text{sat2}}(r_2 - 1)
\]

Only the strongest mode survives

\[
F_{\text{sat1}}(r_1 - 1) \\
F_{\text{sat2}}(r_2 - 1)
\]

\[
\xi_{12} \\
\xi_{21}
\]

\[
0 \\
F_1 \\
F_2
\]

\[
\star = \text{unstable solution} \\
\bullet = \text{stable solution}
\]
Dual-mode class-A laser

Simultaneous oscillation!

Stable if the coupling constant obeys:

\[ C = \xi_{12} \xi_{21} < 1 \]
Dual-mode class-A laser

Bistability!

Stable if the coupling constant obeys:

\[ C = \xi_{12} \xi_{21} > 1 \]
Vectorial bistability and simultaneity of the two helicoidal stationary eigenstates of a ring laser

Fabien Bretenaker and Albert Le Floch

- Two polarization modes
- Cavity length scanned in both directions
Transient behavior of class-B dual-mode laser

- Back to the general equations for the class-B laser:

\[
\begin{align*}
\frac{dF_1}{dt} &= -\frac{F_1}{\tau_{cav1}} + \kappa_1 F_1 \Delta N_1 \\
\frac{d}{dt} \Delta N_1 &= \frac{1}{\tau} (\Delta N_{01} - \Delta N_1) - 2^* \kappa_1 \Delta N_1 (F_1 + \xi_{12} F_2) \\
\frac{dF_2}{dt} &= -\frac{F_2}{\tau_{cav2}} + \kappa_2 F_2 \Delta N_2 \\
\frac{d}{dt} \Delta N_2 &= \frac{1}{\tau} (\Delta N_{02} - \Delta N_2) - 2^* \kappa_2 \Delta N_2 (F_2 + \xi_{21} F_1)
\end{align*}
\]

- Suppose that simultaneous oscillation is stable \((C < 1)\):

\[
\begin{align*}
F_1^0 &= \frac{F_{sat1}(r_1 - 1) - \xi_{12} F_{sat2}(r_2 - 1)}{1 - \xi_{12} \xi_{21}} \\
\Delta N_1^0 &= \frac{1}{\kappa_1 \tau_{cav1}} \\
F_2^0 &= \frac{F_{sat2}(r_2 - 1) - \xi_{21} F_{sat1}(r_1 - 1)}{1 - \xi_{12} \xi_{21}} \\
\Delta N_2^0 &= \frac{1}{\kappa_2 \tau_{cav2}}
\end{align*}
\]
**Transient behavior of class-B dual-mode laser**

- Let us suppose that the two modes have **same parameters**:
  - **Losses**: \( \tau_{\text{cav}1} = \tau_{\text{cav}2} \equiv \tau_{\text{cav}} \)
  - **Cross-section**: \( \kappa_1 = \kappa_2 \equiv \kappa \)
  - **Pumping**: \( \Delta N_{01} = \Delta N_{02} \equiv \Delta N_0 \)
  - **Cross-saturation**: \( \xi_{12} = \xi_{21} \equiv \xi \)

- leading to identical:
  - **Saturation intensities**: \( F_{\text{sat}1} = F_{\text{sat}2} \equiv F_{\text{sat}} \)
  - **Excitation ratios**: \( r_1 = r_2 \equiv r \)
  - **Steady-state inversions**: \( \Delta N_1^0 = \Delta N_2^0 = \frac{1}{\kappa \tau_{\text{cav}}} \equiv \Delta N_{\text{th}} \)
  - **Steady-state intensities**: \( F_1^0 = F_2^0 = \frac{F_{\text{sat}}(r - 1)}{1 + \xi} \equiv F^0 \)
Coupled mechanical oscillators

Expériences : oscillateurs harmoniques couplés

Mécanique, cours 27.exp
Jean-Philippe Ansermet

Prof. Ansermet’s MOOC, EPFL, Switzerland
Symmetric relaxation mode

- Same fluctuations for the two modes:

\[ F_1(t) = F_2(t) = F^0[1 + x(t)] \]
\[ \Delta N_1(t) = \Delta N_2(t) = \Delta N_{th}[1 + y(t)] \]

- Inject into the laser equations and linearize, leading to:

\[
\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{with} \quad M = -\begin{pmatrix} 0 & 1/\tau_{cav} \\ -(r-1)/\tau & -r/\tau \end{pmatrix}
\]

\[
\lambda_{\pm} = -\frac{r}{2\tau} \pm \frac{i}{2} \sqrt{\frac{4(r-1)}{\tau \tau_{cav}} - \left(\frac{r}{\tau}\right)^2}
\]

\[
f_{\text{relax}} \approx \frac{1}{2\pi} \sqrt{\frac{r-1}{\tau \tau_{cav}}} \]

“Standard” relaxation oscillation frequency
Anti-symmetric relaxation mode

- Opposite fluctuations for the two modes:

\[ F_1(t) = F^0[1 + x(t)] \quad \Delta N_1(t) = \Delta N_{th}[1 + y(t)] \]
\[ F_2(t) = F^0[1 - x(t)] \quad \Delta N_2(t) = \Delta N_{th}[1 - y(t)] \]

- Inject into the laser equations and linearize, leading to:

\[
\frac{d}{dt} \left( \begin{array}{c} x \\ y \end{array} \right) = M \left( \begin{array}{c} x \\ y \end{array} \right) \quad \text{with} \quad M = - \left( \begin{array}{cc} 0 & \frac{1}{\tau_{cav}} \\ -\frac{1-\xi}{1+\xi} \frac{r-1}{\tau} & -\frac{r}{\tau} \end{array} \right)
\]

\[
\lambda_{\pm} = -\frac{r}{2\tau} \pm \frac{i}{2} \sqrt{4 \left( \frac{1 - \xi}{1 + \xi} \right) \left( \frac{r - 1}{\tau \tau_{cav}} \right) - \left( \frac{r}{\tau} \right)^2}
\]

\[
f_{\text{anti}} \sim \frac{1}{2\pi} \sqrt{\left( \frac{1 - \xi}{1 + \xi} \right) \left( \frac{r - 1}{\tau \tau_{cav}} \right)}
\]

“Antiphase” relaxation oscillation frequency
Example: intensity noise in a dual-mode laser

- **Intensity fluctuations** in a two-polarization-mode Er,Yb:Glass microchip laser:

![Diagram of the microchip laser](Image)

Fig. 1. Schematic of the microchip laser.

![Graphs showing intensity noise](Image)

Fig. 3. Polarization-resolved intensity recordings versus (a) time and (b) frequency (resolution bandwidth 3 kHz, averaging time 2.8 s); power spectral densities of $I_x$ and $I_y$ are $-65$ dB/Hz at $\Omega_L$ and $-75$ dB/Hz at $\Omega_R$. 
Dual-mode laser: conclusion

- Dual-mode laser: funny dynamics (nonlinear dynamical system)

- Possibility of bistable operation. Applications to optical memories

- Possibility of controlled simultaneous oscillation of two modes. Applications to remote sensing, microwave photonics, etc...

- Can be extended and generalized to more than two modes
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In *Injection* locking

- Consider a single-frequency, class A laser (slave) injected by another laser (master laser, with frequency $\omega_1$)

- What is the influence of the injected field on the slave laser characteristics?

- Slave laser field expanded at master field frequency:

$$E(z,t) = A(t)e^{-i(\omega_1 t - kz)} + c.c.$$
Rate equation for injection locking

- Denote master laser electric field incident on mirror as $E_1$, with complex amplitude $A_1$. The intracavity field is governed by:

$$\frac{dA}{dt} = -\frac{A}{2\tau_{cav}} \left[ 1 - \frac{r}{1 + \frac{2\varepsilon_0|A|^2}{I_{sat}}} \right] + i\delta_{cav}A + \frac{1}{2\tau_{inj}}A_1$$

With

$$\delta_{cav} = \omega_1 - \omega_q$$
$$\tau_{inj} = \frac{L_{cav,opt}}{c_0} \frac{1}{T_1}$$

- Without injection

$$A(t) = |A(t)| \exp(i\varphi) \exp(-i(\omega_q - \omega_1)t)$$

- Introduce amplitude and phases of the fields

$$A = |A|e^{i\varphi}$$
$$A_1 = |A_1|e^{i\varphi_1}$$
Rate equation for injection locking

• Derivative in terms of phase and amplitude

\[
\frac{dA}{dt} = e^{i\varphi} \left( \frac{d|A|}{dt} + i\varphi |A| \right)
\]

• The real and imaginary parts of the equation of evolution of the field are given by:

\[
\frac{d|A|}{dt} = -\frac{|A|}{2\tau_{\text{cav}}} \left[ 1 - \frac{r}{1 + \frac{2c\Omega_0|A|^2}{I_{\text{sat}}}} \right] + \frac{1}{2\tau_{\text{inj}}} |A_1| \cos(\varphi_1 - \varphi)
\]

\[
\frac{d\varphi}{dt} = \omega_1 - \omega_q + \frac{1}{2\tau_{\text{inj}}} \left| \frac{A_1}{A} \right| \sin(\varphi_1 - \varphi).
\]
Suppose the master laser field amplitude $A_1$ is weak, the steady-state field amplitude $A_0$ is given by

$$A_0^2 = \frac{I_{\text{sat}}}{2c_0\varepsilon_0} (r - 1)$$

The phase evolution becomes

$$\frac{d\varphi}{dt} = \omega_1 - \omega_q + \frac{1}{2\tau_{\text{inj}}} \frac{|A_1|}{A_0} \sin(\varphi_1 - \varphi)$$

Suppose the phase $\varphi_1 = 0$, and introduce the lock-in frequency defined by

$$\omega_L = \frac{1}{2\tau_{\text{inj}}} \frac{|A_1|}{A_0}$$

Leading to Adler’s equation:

$$\frac{d\varphi}{dt} = \omega_1 - \omega_q - \omega_L \sin(\varphi)$$
Locked solution

• In the case where $|\omega_1 - \omega_q| < \omega_L$, Adler’s equation has the following steady-state solution

$$\sin(\varphi) = \frac{\omega_1 - \omega_q}{\omega_L}$$

• The frequency (and phase) of the slave is locked to that of the oscillator. The master laser has transferred its frequency to the slave.

• The detuning width of this locking region is given by $2\omega_L$. This locking range is proportional to the square root of master laser intensity.
• In the case where $|\omega_1 - \omega_q| > \omega_L$, Adler’s equation cannot exhibit steady-state solutions. The phase difference between slave and master evolves with time, leading to a beatnote frequency. Rewrite Adler’s equation

$$\frac{d\varphi}{1 - \frac{\omega_L}{\omega_1 - \omega_q} \sin(\varphi)} = (\omega_1 - \omega_q)dt$$

• To find the oscillation period of $\varphi$, integrate between $\varphi = 0$ and $\varphi = 2\pi$.

$$\int_0^{2\pi} \frac{d\varphi}{1 - \frac{\omega_L}{\omega_1 - \omega_q} \sin(\varphi)} = (\omega_1 - \omega_q)T$$
Beat-note frequency

- Use the following integral, valid for $|a| < 1$

$$\int_0^{2\pi} \frac{d\varphi}{1 - a \sin(\varphi)} = \frac{2\pi}{\sqrt{1 - a^2}}$$

- Leading to

$$T = \frac{2\pi}{\sqrt{(\omega_1 - \omega_q)^2 - \omega_L^2}}$$

$$f_{\text{beat}} = \frac{1}{2\pi} \sqrt{(\omega_1 - \omega_q)^2 - \omega_L^2}$$
We have only derived the period of oscillation of the relative phase, not its full evolution.

Numerical solution of the equation leads to the full evolution of the relative phase.

- **Locked case:** $\varphi$ is fixed.
- **Free-running case:** $\varphi$ evolves quasi linearly.
- **In between:** anharmonic evolution.