Experimental investigation of deterministic and stochastic frequency noises of a rapidly frequency chirped laser

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Abstract. The different types of errors affecting the spectral purity of a rapidly chirped continuous wave laser are experimentally studied using an unbalanced interferometer. The response of the interferometer to the deterministic (periodic and aperiodic) and stochastic quantum and technical laser frequency noises is theoretically analysed and experimentally investigated with an external cavity diode laser operating at 793 nm and providing chirps in excess of 10 GHz on the ms timescale. The system is shown to be able to measure the laser frequency to better than 1 MHz during the chirp. It provides an error signal that can be used to servo-control the laser frequency in real time while it is chirped.

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1 Introduction

Monomode laser sources with fast frequency chirping capabilities are mandatory in several fields including optical processing of RF signals [1, 2], optical data storage [3, 4], coherent laser radar [5], more recently in optical true time delay generation [6], and coherent manipulation of atoms [7]. All these fields need fast and mode-hop-free frequency scans through a broad spectral range with a high spectral purity [8–10]. In RF signal processing experiments using rare earth ion doped crystals [1] for instance, one aims at analysing RF signals over a bandwidth of 10 GHz with a resolution of the order of 1 MHz. Several architectures are currently investigated to reach such performances. Some consist in spatially or temporally diffracting the frequency components of the light carrying the considered RF signal on a spectral grating engraved in the populations of the rare earth ion. Others consist in engraving the RF signal to be analysed in the populations of the ions and to read the resulting spectrum with a chirped laser. In both cases, the duration of the analysis must be shorter than the excited level population decay time. This requires that the optical frequency scan (typically 10 GHz) be faster than this lifetime, which is typically in the ms range. With such broad scans and large chirp rates, high linearity and reproducibility are mandatory to preserve the spectrum analyser resolution of 1 MHz. This means that the laser frequency deviation from a perfectly linear chirp has to be lower than 1 MHz. In order to fulfil this condition, a complete measurement of laser frequency noises during the chirp is necessary. Two different families of phase errors occurring during the laser chirp can be distinguished. First, deterministic errors, such as, e.g., the non-linearities of the chirp or some extra frequency modulations have to be isolated. Second, stochastic errors, such as the phase random walk induced by the spontaneous emission or the technical 1/f frequency noise, have to be characterised. These noises exist of course in the case of fixed frequency lasers and can be measured with well-known techniques [11–13]. The problem here is to measure these noises while the average frequency is rapidly swept.

Some experiments dealing with the phase noise characterisation of frequency-chirped CW lasers have already been reported [5, 14–17]. In reference [14], a swept Fabry-Perot étalon allows a determination of the average value of the chirp rate of the laser. However, this can be used only for relatively small and slow chirps and does not investigate the stochastic part of the laser frequency noise. In references [15] and [16], a self-heterodyne technique has been developed to measure and compensate for some incoherent parts of the chirped laser spectrum. This technique is efficient to suppress some superimposed laser frequency modulation but does not permit to observe all sources of frequency deviations in the whole relevant bandwidth. In references [5] and [17] finally, interferometric techniques are used to control the frequency excursion of frequency-modulated lasers in FMCW (Frequency Modulation Continuous Wave) range measurements. Reference [17] deals with chirp rates and amplitudes much lower than our needs. In reference [5], a fiber-based interferometer is developed to observe the laser frequency noise along several periods of frequency modulation, and in
particular to compensate for the nonlinear frequency response of the current-modulated laser diode. Consequently, to our best knowledge, no experiment has been developed for the complete characterisation of single linear frequency chirps in the range necessary for RF signal processing (10 GHz in 1 ms) with the required resolution (1 MHz).

In this paper, we thus investigate a self-heterodyne technique that allows full characterisation of a chirped laser. We apply this technique to a homemade external cavity diode laser (ECDL) rapidly chirped through several free spectral ranges [9, 10]. By this way, we study both the deterministic and stochastic errors that spoil the linearity and the reproducibility of the laser chirps. The way the experimental signals are analysed is specific to our case of a rapidly frequency-swept laser.

This paper is organised as follows. In Section 2, we describe theoretically the chosen method and its capability to give us a signal for characterising the frequency-chirped response and the noises that can occur. Then, in Section 3, we describe the system and the experiments to characterise the deterministic and stochastic frequency errors. Section 4 contains concluding remarks.

2 Principle of operation and predictions

For a chirped laser operating above threshold with constant amplitude, the phase fluctuations are the major causes of spectral impurity. A well-known process to detect the phase fluctuations of usual, i.e., fixed frequency lasers is provided by path-difference interferometers. This is why we choose here also a self-heterodyne interferometry technique for characterising the deterministic and stochastic phase errors of a rapidly chirped laser. This consists in an unbalanced Mach-Zehnder interferometer with a time delay \( \tau_d \) between the two paths, as schematised in Figure 1. In this section, we discuss the capabilities of such a system to characterise linear frequency sweeps without and with phase fluctuations. We neglect amplitude fluctuations.

2.1 Principle of operation without phase noises

At the laser output and in the absence of noise, the electrical field can be expressed as:

\[
E(t) = \mathcal{E}(t) + cc = E_0 \exp \left[ j2\pi \left( \nu_0 t + \frac{r}{2} t^2 \right) + j\phi_0 \right] + cc,
\]

where \( E_0 \) is the light field amplitude which we suppose constant, \( \nu_0 \) is the average optical frequency, \( r \) is the chirp rate value and \( \phi_0 \) is a constant phase. In the unbalanced Mach-Zehnder interferometer schematised in Figure 1, the laser beam is split in two arms. One of them experiences a delay \( \tau_d \) while the other is frequency shifted by an acousto-optic modulator operating at frequency \( \Delta \nu \). If the contrast is supposed to be equal to 1, the recombination of these two beams at the output of the interferometer leads to the following field:

\[
E_T(t) = \frac{1}{\sqrt{2}} \left[ \mathcal{E}(t) \exp(-j2\pi \Delta \nu t) + \mathcal{E}(t - \tau_d) + cc \right].
\]

Consequently, the detected intensity is

\[
I(t) = 2E_0^2 \left[ 1 + \cos \left( 2\pi f_b t + 2\pi \nu_0 \tau_d - \pi r \tau_d^2 \right) \right],
\]

where \( f_b = r\tau_d - \Delta \nu \) is the beat signal frequency. We notice that \( f_b \) is a linear function of the chirp rate \( r \). The analysis of the laser chirp rate \( r \) can consequently be performed by a simple Fourier analysis of the beat note signal. Of course, the typical resolution of the resulting spectrum will be of the order of \( 1/T \), where \( T \) is the duration of the considered chirp. However, the precision on the measurement of \( r \) can be adjusted by modifying the delay \( \tau_d \). Moreover, even for low values of \( \tau_d \), the signal can be shifted out of the low frequency noise region to higher frequencies thanks to the acousto-optic modulator frequency \( \Delta \nu \). Depending on the nature of the phase errors that are predominant, different measurement strategies can be implemented, as we are going to see now.

2.2 Influence of deterministic frequency errors

In the following, we consider different sources of phase fluctuations distorting the laser chirp purity and we see how our interferometer can detect them. These phase fluctuations will be introduced in the laser field of equation (1) by replacing \( \phi_0 \) by a time varying function \( \phi(t) \). We distinguish between two main kinds of phase errors \( \phi(t) \). In this subsection, we show how our system can detect and characterise deterministic errors, such as an extra frequency modulation occurring during the chirp or a nonlinearity of the chirp. The case of stochastic phase errors will be discussed in the following subsection.

First, let us suppose that the chirped laser is affected by an extra frequency modulation. We represent it by the phase modulation \( \phi(t) = \frac{2\pi}{T} \sin(2\pi F t) \), where \( \delta \nu_{\text{mod}} \) and \( F \) are respectively the amplitude and the frequency of the frequency modulation. Such a variation of the laser frequency can be induced by a variation of the laser losses, as we will see in the experimental section of this paper.
Then the detected intensity of equation (3) becomes:

\[ I (t) = 2E_0^2 \left\{ 1 + \cos \left[ 2\pi f_0 t + \frac{2\nu t + 2\alpha \tau_d^2}{F} \right] \times \sin (2\pi F t - \pi \tau_d t) \cos (\pi F \tau_d + \psi_0) \right\}, \]

where \( \psi_0 = 2\pi \nu_0 \tau_d - \pi \tau_d^2, \phi' = \pi F \tau_d, \) and \( A = \frac{2\nu}{F} \cos (\phi''). \) Using an expansion of the cosine of equation (4) in terms of Bessel functions, we can see that the beatnote signal is transformed into a typical FM signal:

\[ I (t) = 2E_0^2 \left\{ 1 + J_0 (A) \cos (2\pi f_0 t + \psi_0) \right. \\
+ \sum_{p=1}^{\infty} J_{2p} (A) \left[ \cos (2\pi (f_0 + 2pF) t + \psi_0 + 2p\phi') \right. \\
+ \cos (2\pi (f_0 + 2pF) t + \psi_0 - 2p\phi') \right. \\
+ \sum_{p=0}^{\infty} J_{2p+1} (A) \left[ \cos (2\pi (f_0 - (2p + 1) F) t + \psi_0 \right. \\
- (2p + 1)\phi') - \cos (2\pi (f_0 + (2p + 1) F) t + \psi_0 \\
- (2p + 1)\phi') \right] \right\}, \]

where \( p \) is an integer. We can thus see that the spectrum of the perfectly chirped laser is enriched by the presence of sidebands at multiples of the modulation frequency \( F. \) In particular, in the case where the frequency modulation amplitude is weak, all these sidebands except the ones at frequencies \( f_0 \pm F \) can be neglected and the relative amplitude \( J_1 (A)/J_0 (A) \approx A/2 \) of these sidebands can be used to determine the amplitude of the frequency modulation, as we will see later experimentally.

Let us now suppose that the laser frequency chirp is affected by non-linearities. We assume for example that the laser instantaneous frequency is given by \( \nu = \nu_0 + \nu t + \alpha \tau_d^2, \) where \( \alpha \) is the lowest order non-linear term. Consequently, the detected optical intensity is

\[ I (t) = 2E_0^2 \left[ 1 + \cos \left\{ 2\pi \left( f_0 - \alpha \tau_d^2 + \alpha \tau_d t \right) t + \psi_0 + \frac{2\pi \alpha \tau_d^2}{3} \right\} \right]. \]

In this case, it clearly appears that the beatnote frequency is time dependent and can be expressed as \( f (t) = f_0 - \alpha \tau_d^2 + 2\alpha \tau_d t. \) Therefore, if we can measure a variation of the beatnote frequency during the chirp, we can conclude that the laser chirp is not linear. The precision on the measurement of this nonlinearity is of course related to the value of the delay \( \tau_d \) and to the duration \( T \) of the chirp which determines the resolution of the spectral analysis of the experimental signal.

### 2.3 Influence of stochastic frequency noises

Let us now turn to the case of stochastic frequency noises that can affect the spectral purity of chirped lasers. We can expect our laser to be affected by two main types of frequency noises: (i) the white frequency noise due to spontaneous emission which leads to the Schawlow-Townes linewidth in the case of usual stable frequency lasers and (ii) the low frequency (typically \( 1/T \)) technical frequency noise induced by mechanical or acoustic variations of the laser cavity length.

In both cases, we can write the instantaneous phase of a linearly chirped laser as

\[ \Phi (t) = 2\pi \left[ \nu_0 t + \frac{\nu t^2}{2} \right] + \phi (t). \]

Here, \( \phi (t) \) can represent both the white quantum and the low-frequency technical noises. At the output of the interferometer the optical intensity incident on the detector can be written as

\[ I (t) = 2E_0^2 \left( 1 + \cos \left[ 2\pi f_0 t + \psi_0 + \phi (t) - \phi (t - \tau_d) \right] \right). \]

A sine wave is observed at the output of the photo detector with an average frequency \( f_b. \) To obtain the spectrum of this signal, we need to calculate the auto-correlation function of \( I (t) \) and apply the Wiener-Khintchine theorem. The auto-correlation function is given by

\[ R_I (t, \tau) = \langle I (t) I (t + \tau) \rangle, \]

where the brackets \( \langle \rangle \) denote ensemble averaging. This leads to

\[ R_I (t, \tau) = 4E_0^4 + 2E_0^4 \cos \left( 2\pi f_b \tau + \phi (t) \right) \cos \left( 2\pi f_b \tau + \phi (t - \tau_d) \right), \]

where we have defined

\[ H (t, \tau) = \phi (t + \tau) - \phi (t) - \phi (t + \tau - \tau_d) + \phi (t - \tau_d). \]

\( H (t, \tau) \) depends on the optical phase at four different times. We suppose that the phase jitter defined by

\[ \Delta \phi (t, \tau) = \phi (t + \tau) - \phi (t) \]

is a zero-mean stationary Gaussian process. Then the dependence in \( t \) disappears in the moments of \( H (t, \tau), \) and the auto-correlation function of the optical intensity can be written

\[ R_I (t, \tau) = 4E_0^4 + 2E_0^4 \exp \left\{ - \frac{\langle H^2 (t, \tau) \rangle}{2} \cos (2\pi f_b \tau) \right\}. \]

The Fourier transform of \( R_I (t) \) leads to the spectrum of the intensity detected at the output of the interferometer. Let us first suppose that the only noise occurring in the frequency chirped laser is the white quantum noise. In this case, the auto-correlation of the frequency error is given by:

\[ \langle \dot{\phi} (t + \tau) \dot{\phi} (t) \rangle = \frac{1}{\tau_c} \delta (\tau), \]

where \( \tau_c \) is the laser coherence time. From equations (11, 12), one can express \( R_I (t) \) as a function of the phase jitter by writing as:

\[ H (t, \tau) = \Delta \phi (t, \tau) - \Delta \phi (t - \tau_d, \tau). \]
Using equation (14), this leads to:

\[ \langle H^2(t, \tau) \rangle = \begin{cases} 2|\tau|/\tau_c & \text{for } |\tau| < \tau_d \\ 2\tau_d/\tau_c & \text{for } |\tau| > \tau_d. \end{cases} \]  

(16)

This can be used together with equation (13) to obtain:

\[ R_I(\tau) = 4E_0^4 + 2E_0^4 \exp(-\tau_d/\tau_c) \cos(2\pi f_0 \tau) \quad \text{for } |\tau| < \tau_d. \]  

(17)

Consequently, the power spectral density of the detected signal is given by the Fourier transform of equation (17):

\[ S_I(\omega) = 4E_0^4 \delta(\omega) + E_0^4 \exp(-\tau_d/\tau_c) \delta(\omega - 2\pi f_0) \]  

\[ + E_0^4 \exp(-\tau_d/\tau_c) \frac{\tau_c}{\pi} \left[ \exp(\tau_d/\tau_c) - \cos(\omega - 2\pi f_0) \tau_d - \frac{\sin(\omega - 2\pi f_0) \tau_d}{\omega - 2\pi f_0} \right] + \{f_0 \rightarrow -f_0\}. \]  

(18)

This result is similar to the self-heterodyne spectrum obtained for laser sources operating at fixed frequency [11–13], except for the frequency shift \( f_0 \) which varies with the chirp rate. Consequently, the quantum frequency noise of our chirped laser can be obtained as usually by taking \( \tau_d \) as long as possible in order to favour the Lorentzian part of equation (18), which is just the image of the Lorentzian Schawlow-Townes broadening, with respect to the Dirac term.

Let us now turn to the case of a low-frequency noise affecting the laser frequency, induced for example by mechanical or acoustic noise. We suppose in the following calculation that this noise is the predominant process, with respect to which all other noises can be neglected. If we choose the interferometer delay \( \tau_d \) short enough with respect to the characteristic time of the variations of the laser frequency noise \( \dot{\phi}(t) \), i.e., with respect to the laser frequency noise coherence time, the following simplification can be used:

\[ \phi(t) - \phi(t - \tau_d) \approx \tau_d \dot{\phi}(t). \]  

(19)

This can be used to simplify equation (11), leading to:

\[ H(t, \tau) = \tau_d \left( \dot{\phi}(t + \tau) - \dot{\phi}(t) \right). \]  

(20)

To obtain the auto-correlation function of the detected intensity, one needs to calculate the variance of \( H \):

\[ \langle H^2(t, \tau) \rangle = 2\tau_d^2 \left[ \sigma_{\dot{\phi}}^2 - R_{\dot{\phi}}(\tau) \right], \]  

(21)

where \( R_{\dot{\phi}}(\tau) \) is the auto-correlation function of the frequency noise \( \dot{\phi}(t) \) and where we have supposed that the variance \( \sigma_{\dot{\phi}}^2 \) of \( \dot{\phi}(t) \) is finite. In the case of \( 1/f \) noise, this latter condition can be easily fulfilled by bounding the frequency noise spectrum to a finite interval \([\omega_{\text{min}}, \omega_{\text{max}}]\) with \( \omega_{\text{min}} > 0 \). Equation (13) then leads to:

\[ R_I(\tau) = 4E_0^4 + 2E_0^4 \exp \left[ -\tau_d^2 \sigma_{\dot{\phi}}^2 \right] \times \exp \left[ \tau_d^2 R_{\dot{\phi}}(\tau) \right] \cos(2\pi f_0 \tau). \]  

(22)

An analytical expression of the spectrum can be obtained if \( \tau_d \) is short enough to fulfill the following condition:

\[ \tau_d^2 R_{\dot{\phi}}(\tau) \leq \tau_d^2 \sigma_{\dot{\phi}}^2 \ll 1. \]  

(23)

Then the exponential term \( \exp(\tau_d^2 R_{\dot{\phi}}(\tau)) \) in equation (22) can be expanded to first order. The Fourier transform of equation (22) can then be calculated analytically, leading to:

\[ S_I(\omega) = 4E_0^4 \delta(\omega) + E_0^4 \exp \left[ -\tau_d^2 \sigma_{\dot{\phi}}^2 \right] \times \left[ \delta(\omega - 2\pi f_0) + \tau_d^2 S_{\dot{\phi}}(\omega - 2\pi f_0) \right] + \{f_0 \rightarrow -f_0\}. \]  

(24)

This equation shows that the interferometer has transferred the low-frequency frequency noise at the foot of the Dirac peak created by the beatnote at \( f_0 \). This low-frequency noise should consequently easily be observed with this interferometer.

Of course, in the general case, the quantum white frequency noise and the technical low-frequency noise will coexist. However, as we will see in the experimental section of this paper, a careful choice of the interferometer delay \( \tau_d \) can allow one to combine the results of equations (18) and (24), and hence to observe both noise components simultaneously.

3 Experimental results

3.1 Description of the experimental setup

Figure 2 is a schematic of the experimental setup used to characterise both the deterministic and stochastic noises degrading the chirped laser spectral purity. The laser under test is the one we use to generate chirps in our experiments on the spectral analysis of RF signals using the optical transition of \( \text{Tm}^{3+} \) ions at 793 nm [1]. It is an external cavity diode laser (ECDL) closed by a grating in Littrow configuration. It contains a prismatic \( \text{LiTaO}_3 \) electro-optic crystal which turns the laser into a voltage-controlled oscillator, able to generate continuous frequency chirps over several free spectral ranges without mode hops [9, 10]. This voltage-controlled oscillator is characterised by a scale factor \( K = 80 \text{ V/GHz} \) and a 2 MHz bandwidth, limited by the piezoelectric resonances of the crystal. The laser frequency can hence routinely be scanned over several GHz within a \( \mu \text{s} \). The output from this laser is directed through amorphous prisms immediately followed by an optical isolator. To build our unbalanced interferometer, we then
Fig. 2. Experimental setup. ECDL: external cavity diode laser containing the electro-optic crystal EO; FP: Fabry-Perot analyser; AMP: high-voltage amplifier; HWP: half-wave plate. PD1, PD2: 100-MHz bandwidth photodiodes.

split the beam into two paths using a 50/50 beamsplitter. A portion of one beam is monitored with a scanning Fabry-Perot interferometer to make sure that the laser is operating in a single, stable longitudinal mode. The remainder of this beam is delayed by an interchangeable length of single-mode optical fibre. The light travelling through the other arm is frequency shifted by an acousto-optic modulator operating at 80 MHz. Depending on the experiment, the acousto-optic modulator can be used or not. The two beams are then recombined by another beamsplitter. The two channels are detected by photodetectors with a 100 MHz bandwidth. We use non-polarisation maintaining fibre. Consequently, the fibre coil behaves as the combination of a linear birefringent element of arbitrary retardance and orientation with a rotation. To compensate for the slow variations of this birefringence, we use two half-wave plates and two polarisers, as shown in Figure 1. These components permit to control the phase difference between the signals detected by the two photodiodes. These signals are then recorded in a fast oscilloscope with a large memory before further analysis.

Fig. 3. (a) Typical experimental signal obtained for a voltage variation of 100 V in 100 ms with \( \tau_d = 33.3 \) ns. (b) Evolution of the observed beat frequency versus \( r \).

Fig. 4. Experimental spectra obtained with a delay \( \tau_d = 33.3 \) ns for a chirp duration of (a) 100 ms and (b) 1 ms with a chirp rate (a) \( r = 11.7 \) GHz/s and (b) \( r = 11.5 \) THz/s.

3.2 Measurement of deterministic frequency errors

Two typical results of FFT analysis of experimental records are reproduced in Figure 4, for the two extreme values of \( r \) displayed in Figure 3b. Obviously, the precision of the measurement of the beat frequency is given by the inverse \( 1/T \) of the duration of the chirp. Thus, since \( f_b \) is a linear function of \( r \), the precision on the measurement of the chirp value is \( \Delta r = 1/T\tau_d \). This corresponds to the widths of the main peaks of the spectra of Figures 4a and 4b, leading to a precision \( \Delta r \) equal to 300 MHz/s and 30 GHz/s respectively. This precision on the measurement of \( r \) corresponds to a maximum error on the measurement of the laser frequency \( \Delta \nu_{\text{max}} = T\Delta r = 1/\tau_d \). This corresponds to the interval between two successive fringes of our interferometer. In the case of Figure 4, it is equal to 30 MHz. This is well above the precision required for our RF spectral analysis applications (better than 1 MHz resolution). To improve this precision, one can either increase \( \tau_d \) or implement a detection scheme able to detect dephasings much smaller than \( 2\pi \). Both directions will be investigated below.

For now, even our relatively low value of \( \tau_d \) permits to isolate and measure other errors occurring during the chirp. Indeed, the two spectra of Figure 4 present some
have checked for different values of modulation of the polarisation of the laser mode, which corresponds to a variation of \( \Delta \nu \) during one period of this modulation. Contrary, the one of Figure 4b must be attributed to the laser due to an electrical or mechanical vibration. On the contrary, the one of Figure 4b must be attributed to the laser itself. Indeed, one can check that the laser frequency variation occurring during one period of this modulation corresponds to a variation of \( 2\pi \) of the retardance of the intra-cavity electro-optic crystal. Together with the existence of a small misalignment of the polarisation of the laser diode with respect to the \( z \) axis of the crystal, this leads to a modulation of the polarisation of the laser mode, which creates a modulation of the laser losses and frequency. We have checked for different values of \( r \) and \( \Delta \nu_{\text{laser}} \) that the period of this spurious modulation is given by \( 2V_{\pi}/V \), where \( V_{\pi} \) is the voltage creating a \( \pi \) retardation when applied to the crystal and \( V \) is the voltage ramp slope. Of course, we have also checked that this modulation is visible only when the amplitude of the ramp is larger than \( 2V_{\pi} \), i.e., when at least one period of modulation is present.

As we have mentioned above, to improve the resolution on the laser frequency error measurement, we have to increase the delay of the interferometer. We thus now choose \( \tau_d = 150 \) ns. We thus expect the frequency resolution of our measurement to be of the order of a few MHz, hence giving us access to new types of deterministic laser frequency errors. This is illustrated in Figure 5. Figure 5a corresponds to the FFT of the signal obtained for a chirp amplitude \( \Delta \nu_{\text{laser}} = 10 \) GHz in a duration \( T = 20 \) ms. The width of this spectrum is much broader (of the order of a few kHz) than what could be expected from the resolution of the FFT (\( 1/T = 50 \) Hz), indicating an important deviation of the laser instantaneous frequency with respect to a perfectly linear chirp. To analyse this deviation, one can divide the 20 ms-long signal in ten successive records lasting 2 ms each. The FFT spectra of these ten successive records are shown in Figure 5b. As can be seen in Figure 5c, the average beat frequency \( f_b \) increases quasi-linearly as a function of time during the 20 ms-long chirp. Consequently, the laser instantaneous frequency can be modelled by the expression \( \nu(t) = \nu_0 + rt + \alpha t^2 \), with \( r = 0.47 \) THz/s and \( \alpha = 1.5 \) THz/s². This error, which is probably due to an imperfection of the diode laser anti-reflection coating, has been isolated and measured after the chirp is finished. A more useful technique would consist in measuring the chirp errors in real time. This will be the subject of Section 3.4.

### 3.3 Measurement of stochastic frequency noises

Let us now investigate whether our experimental setup is able to measure stochastic frequency errors occurring during the chirps. We have seen in Section 2.3 that the laser white frequency noise induced by spontaneous emission can be observed in the very same conditions as for a stable frequency laser, except for a centre frequency offset due to the chirp. However, the power spectral density of equation (18) contains two terms centred at the beatnote frequency \( f_b = \nu_0 - \nu_c \), so that the self-heterodyne spectrum of equation (18) becomes a simple Lorentzian twice as large as the laser spectrum [11, 12]. However, here, we expect \( \tau_c \) to be of the order of a few tens of microseconds. The condition \( \tau_d \gg \tau_c \) would require the use of several kilometres of monomode optical fibre, which is very expensive and rather lossy at our wavelength of 793 nm. This is why we use here only about 400 m of fibre, leading to a delay \( \tau_d \approx 2.1 \) ms. Let us first check that even with this relatively short delay we can obtain a good measurement of the laser frequency white noise.

To check this, we first keep the laser frequency constant \( (r = 0) \) and we introduce the acousto-optic modulator at 80 MHz. The self-heterodyne spectrum around 80 MHz is observed using a spectrum analyser. Two typical spectra obtained for two different values of the laser diode current are displayed in Figure 6a. For the sake of clarity we have removed the peak at 80 MHz due to the Dirac term in equation (18). As expected, we obtain an oscillating spectrum with a period equal to \( 1/\tau_d = 480 \) kHz. The envelope and the contrast of these oscillations depend on the value of \( \tau_c \). These data are fitted using the expression of equation (18), leading to the thick lines of Figure 6a. The two fits displayed in Figure 6a correspond to \( \tau_c = 21.5 \) ms and \( \tau_c = 16.2 \) ms. As expected, the measured coherence time \( \tau_c \) increases with the laser output power, as can be seen from the measurements reproduced in Figure 6b. The solid line is a linear adjustment which is consistent with what is expected from calculations of
the Schawlow-Townes linewidth of our laser, taking the parameters of the cavity and of the active medium into account [18–20].

The results of Figure 6 show the ability of our system to measure the white noise component of the laser frequency. We now turn to the real aim of the experiment, which is to see whether the level of this white noise is modified when the laser frequency is rapidly chirped. However, in this case, due to the relatively short duration of the laser chirp, we can no longer use the spectrum analyser to obtain the self-heterodyne spectrum. We have to record the whole signal and process it using a FFT algorithm, as above. The result of such a measurement is shown in Figure 7 for three different values of \( r \) and for \( T = 1 \) ms. One can first check that the value of \( f_b \) decreases with \( r \), as expected. Besides, due to the fact that we can no longer use the spectrum analyser, the results are much noisier than those of Figure 6a, due to the digitalisation noise of our oscilloscope. However, two conclusions can be drawn from these results. First, when we fit the oscillating part of the spectra of Figure 7 with the expression of equation (18), we obtain the same result for \( \tau_c \) as that we then obtain with a resolution bandwidth of 1 kHz, leading to the spectrum of Figure 8b. Moreover, we see that the level of this white noise is due to the spontaneous emission of the active medium. However when \( r \) is increased, one can see from the decrease of the signal-to-noise ratio of the spectra of Figure 7, and also from the fact that the noise level close to the centre frequency increases, that another component of the laser frequency noise becomes significant when the laser is chirped.

This noise component is probably a low-frequency component of the laser frequency noise. However, we have seen in Section 2.3 that to measure the spectrum of such a frequency noise, we must choose the delay \( \tau_d \) of the interferometer much shorter than the typical timescale of the variations of the instantaneous laser frequency. The value \( \tau_d = 2.1 \) \( \mu \)s used to obtain the results of Figure 7 is clearly too long. This is why we turn again to a rather small interferometer path difference using a 20 m-long fibre, leading to \( \tau_d = 100 \) ns. Contrary to the results of Figures 3–5 which led to the measurement of deterministic frequency errors, we keep the acousto-optic frequency shifter at 80 MHz, in order to shift the spectrum far from the zero frequency. In order to keep a sufficient measurement dynamics, we have to analyse the signal using the spectrum analyser. The resolution we want (1 kHz) obliges us to work with very slow chirps only (\( T = 2.5 \) s).

Figure 8a shows the typical spectra (here with \( r = 0 \)) that we then obtain with a resolution bandwidth of 30 kHz. Three different parts can be distinguished in this spectrum: (i) the Dirac term at \( f_b \); (ii) the oscillations due to the white frequency noise and given by equation (18), that are consistent with the value \( \tau_c = 20 \) \( \mu \)s determined from the preceding measurements; and (iii) a broadening, together with extra lines at the foot of the Dirac peak, over a bandwidth of the order of 500 kHz. This last low-frequency component is the one we wish to investigate. As required [see the discussion of Eq. (19)], its bandwidth is much smaller than \( 1/\tau_d = 10^7 \) s\(^{-1}\). We hence zoom on a 2 MHz bandwidth around 80 MHz with a resolution bandwidth of 1 kHz, leading to the spectrum of Figure 8b. We can clearly see the extra noise component lying at the
foot of the Dirac peak and on top of the floor noise given by the white noise component of the self-heterodyne spectrum. We use equation (24) to extract the low-frequency component of the power spectral density $S_{\nu\nu}(\omega)$ of the instantaneous frequency error $\delta\nu = \phi/2\pi$. This leads to the spectrum of Figure 8c. This noise is quite well fitted by a $1/f$ law between 5 kHz and 400 kHz, suggesting a technical origin. The total power of this noise, which gives the variance $\sigma_{\delta\nu}$ of the frequency due to this noise component, is given by the area below the spectrum of Figure 8c and is of the order of 18 kHz. This shows that $\tau_{d}\sigma_{\delta\nu} = 2\pi\tau_{d}\sigma_{\delta\nu} \approx 10^{-2} \ll 1$, as required to derive equation (24). The peak at 864 kHz should not be attributed to the laser frequency: it is due to the ‘Radio Bleue’ emitter of Villebon-sur-Yvette, close to our laboratory. Now, when we turn on the high voltage amplifier at the input of the electro-optic crystal to apply a very slow chirp $r = 1.5$ MHz/ms ($\Delta\nu_{\text{laser}} = 3.75$ GHz in $T = 2.5$ s) to our laser, the central part of the self-heterodyne spectrum becomes the one of Figure 8d. After transformation using equation (24), we then obtain the power spectral density of the instantaneous frequency noise of Figure 8e. The noise is larger than in the case of Figure 8c, with a variance $\sigma_{\delta\nu}$ of the order of 60 kHz. This is still consistent with the hypothesis $\tau_{d}\sigma_{\delta\nu} \ll 1$. The increase of the low-frequency noise has been observed to be independent of the value of $r$, in the range of small values of $r$ achievable with the present experiment. Actually, the extra noise observed in Figure 8e with respect to Figure 8c is due to the high-voltage amplifier which amplifies the ramp applied to the electro-optic crystal. This illustrates the high sensitivity of the present setup.

3.4 Towards a servo-control of the chirped laser frequency

Up to now, the Fourier analysis of the signals delivered by the interferometer has allowed us to measure the different noises affecting the frequency of our chirped laser, but only a posteriori. The orders of magnitude of the different errors we have observed permit us to conclude that if we want to reach a precision better than, say, 1 MHz over the laser frequency during the chirp, we have to take care of the possible deterministic frequency errors, either periodic or not, and of the low-frequency component of the stochastic frequency noise. The white frequency noise has been shown to lead to a Lorentzian broadening of the order of 10 kHz and can be ignored. Consequently, all the frequency noises we have to deal with for our RF spectral analysis applications using rare earth ion doped crystals exhibit bandwidths lying well below 1 MHz. This is within the range of conventional electronic servo control loops and one would be tempted to servo-lock the laser chirp to a perfect linear chirp within an error smaller than 1 MHz. However, to realise this, we need to measure the laser frequency instantaneously. This can be performed using our interferometer, provided that we measure the phase difference $\psi$ between the two arms at the output of the interferometer. This phase can be obtained provided we detect two signals in quadrature at the output of the interferometer [21] using the two photodiodes PD1 and PD2 (see Fig. 2). In general, the phase difference between these two signals and the equality of their amplitudes can be adjusted by tuning the orientations of a quarter-wave plate and a half-wave plate located at the output of the fibre, together with one polariser in front of each detector. Here we simply use one half-wave plate at the output of the fibre. The rotation of this half-wave plate allows us to perfectly adjust the $\pi/2$ dephasing between the two signals, even if their amplitudes cannot be equalised.

Some resulting typical signals are reproduced in Figure 9a. They were obtained with a 20-nm-long fibre, corresponding to $\tau_{d} = 100$ ns, with a chirp rate $r = 1.6$ GHz/ms ($\Delta\nu_{\text{laser}} = 6.4$ GHz in $T = 4$ ns). Since a $2\pi$ dephasing of the interferometer corresponds to a frequency variation of $1/\tau_{c} = 10$ MHz, reaching a precision better than 1 MHz on the laser frequency requires to measure $\psi$ with a resolution better than $\pi/5$. Figure 9a shows that this is easily achieved with the sampling rate of 6.25 MHz that we use. The phase error $\delta\psi(t) = \psi(t) - 2\pi r t \tau_{d}$ induced by the laser frequency error can easily be reconstructed from the two quadrature signals, as evidenced in Figure 9b.
signal contains the different types of errors we have investigated. Indeed, it is the superposition of (i) a quasi-linear drift, (ii) a sinusoidal modulation, and (iii) a stochastic component.

Such a signal could be filtered and used as an error signal to lock the laser frequency in a digital servo loop with a correction voltage applied on the electro-optic crystal, as already used for a fixed frequency laser [22]. As we have seen above, all the errors we have to correct for, i.e., the deterministic frequency errors and the low-frequency part of the stochastic noise, lie in a bandwidth smaller than a few hundreds of kHz. A loop bandwidth of 1 MHz would consequently be sufficient. Besides, each period of the interferogram of Figure 9a corresponds to a variation of the laser frequency of \( \frac{1}{\tau_d} = 10 \text{ MHz} \). Consequently, a resolution better than 1 MHz, as required for spectral analysis applications, could easily be achieved by digitalising the quadrature signals with 8-bit converters. Moreover, the use of a digital servo-loop would permit to easily implement non-linear chirps.

### 4 Conclusion

To conclude, we have shown that an unbalanced interferometer can be used to finely characterise the errors affecting the laser linear frequency chirps necessary for applications to the RF spectral analysis using rare-earth doped crystals. We have indeed proved the ability of this setup to measure deterministic errors, such as extra frequency modulations or drifts of the chirp rate, as well as white or coloured stochastic frequency noises occurring during the chirp with a precision much better than 1 MHz for frequency chirps of the order of 10 GHz in 1 ms. Moreover, we have proved that this setup can be used to build an error signal in order to servo-control the laser frequency chirp in real time during the chirp itself. We have seen that a reasonably achievable 8-bit digital servo-loop with a 1 MHz bandwidth can be imagined. This is the purpose of our present investigations.

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